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Mean-field solution of the random Ising model on the dual lattice

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We perform a duality transformation that allows one to express the partition function of the d-dimensional Ising model with random nearest neighbor coupling in terms of spin variables defined on the square plaquettes of the lattice. The dual model is solved in the mean-field approximation.

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The Ising model with random coupling plays a fundamental role in the theory of disordered systems. In this field, one of the major results is the Parisi solution of the infinite range model where at low temperature the system becomes a spin glass with a replica symmetry breaking [1]. However, there is no exact solution for the model with nearest neighbor interaction, and it is still unclear whereas a glassy phase is present in three dimensions at finite temperature.

In this paper, we perform a duality transformation of the Ising model with random nearest neighbor coupling that assumes the values $J_{ij}=\pm 1$ with equal probability. The model is thus defined on a dual lattice where the spin variables are attached to the square plaquettes. The advantage is that the nonlinear part of the dual Hamiltonian has constant coefficients instead of random ones. It is therefore possible to use the standard mean-field approximation to estimate the quenched free energy. However, in the dual lattice the ratio between number of spins and number of links increases with the dimension in contrast to what happens in the original lattice so that the mean-field approximation becomes worse at increasing the dimensionality. Our solution is thus optimal in two dimensions where it gives a rather good estimate of the ground state energy.

The partition function of the d-dimensional Ising models on a lattice of N sites with nearest neighbor couplings J_{ij} which are independent identically distributed random variables, in the absence of external magnetic field, is

$$Z_N(\beta,\{J_{ij}\})=\sum_{\{\sigma\}}\prod_{(i,j)}\exp(\beta J_{ij}\sigma_i\sigma_j), \tag{1}$$

where the sum runs over the spin configurations $\{\sigma\}$, and the product over the nearest neighbor sites (i,j). One is interested in computing the quenched free energy

$$f=-\lim_{N\to\infty}\frac{1}{\beta N}\overline{\ln Z}, \tag{2}$$

where $\bar{A}$ indicates the average of an observable A over the distribution of the random coupling. The quenched free energy is a self-averaging quantity, i.e., it is obtained in the thermodynamic limit for almost all realizations of disorder [1]. Even in one dimension, it is difficult to find an exact solution for f in the presence of an external constant magnetic field [2]. On the other hand, it is trivial to compute the so-called annealed free energy

$$f_a=-\lim_{N\to\infty}\frac{1}{\beta N}\ln\bar{Z}, \tag{3}$$

corresponding to the free energy of a system where the random couplings are not quenched but can thermalize with a relaxation time comparable to that of the spin variables. An easy calculation shows that in our case

$$f_a=-\frac{1}{\beta}(\ln 2+d\ \ln\cosh\beta). \tag{4}$$

However, f_a is a very poor approximation of the quenched free energy, and is not able to capture the qualitative features of the model.

In order to estimate (1), it is convenient to use the link variable $x_{ij}=\sigma_i\sigma_j$, since only terms corresponding to products of the variables x_{ij} on close loops survive after summing over the spin configurations: on every close loop of the lattice $\Pi x_{ij}=1$, while $\Pi x_{ij}=\sigma_a\sigma_b$ for a path from the site a to the site b. A moment of reflection shows that it is sufficient to fix $\Pi x_{ij}=1$ on the elementary square plaquettes $\mathscr{P}$ to automatically fix it on all the close loops. The partition function thus becomes

$$Z_N(\beta,\{J_{ij}\})=\sum_{\{x_{ij}\}}\prod_{i=1}^{N_p}\frac{1+\tilde{x}_i}{2}\prod_{(i,j)}e^{\beta J_{ij}x_{ij}} \tag{5}$$

where the number of plaquettes is $N_p=d(d-1)N/2$, we have introduced the plaquette variable $\tilde{x}_i=\Pi_{\mathscr{P}_i}x_{ij}$.

For dichotomic random coupling $J_{ij}=\pm 1$ with equal probability, the free energy of the model is invariant under the gauge transformation $x_{ij}\to J_{ij}x_{ij}$, so that one has

$$Z_N=\sum_{\{x_{ij}\}}\prod_{i=1}^{N_p}\frac{1+\tilde{x}_i\tilde{J}_i}{2}\prod_{(i,j)}e^{\beta x_{ij}}, \tag{6}$$

where $\tilde{J}_i=\Pi_{\mathscr{P}_i}J_{ij}$ is again a dichotomic random variable (the "frustration" [3] of the plaquette $\mathscr{P}_i$). It is worth remarking that (6) gives the partition function in terms of a sum over the 2^{dN} configurations of the independent random variables $x_{ij}=\pm 1$ with probability

$$p_{ij}=\frac{e^{\beta x_{ij}}}{2\cosh\beta}. \tag{7}$$

In the following we shall indicate the average of an observable A over such a normalized weight by $\langle A\rangle$, e.g., $\langle x_{ij}\rangle=\tanh\beta$ and $\langle\tilde{x}_i\rangle=\tanh^4\beta$. With such a notation, the partition function assumes the compact form

$$Z_N=2^{(dN-N_p)}\cosh^{dN}(\beta)\left\langle\prod_{i=1}^{N_p}(1+\tilde{x}_i\tilde{J}_i)\right\rangle. \tag{8}$$

Now comes the key step. We estimate the average in (8) by a geometrical construction. Let us introduce the dual lattice [4] as the lattice whose sites are located at the centers of each square of the original lattice. A dual spin variable is attached to each square plaquette and can assume only the values $\tilde{\sigma}_i=\pm1$ with equal probability, so that one has the identity

$$(1+\tilde{x}_i\tilde{J}_i)=\sum_{\tilde{\sigma}_i=\pm1}(\tilde{x}_i\tilde{J}_i)^{(1+\tilde{\sigma}_i)/2}. \tag{9}$$

Since there is a one-to-one correspondence between links on the original and on the dual lattice, we can compute the link average noting that

$$\left\langle\prod_{i=1}^{N_p}\tilde{x}_i^{(1+\tilde{\sigma}_i)/2}\right\rangle=\left\langle\prod_{i,j}x_{ij}^{\Sigma_{k\in(i,j)}(1+\tilde{\sigma}_k)/2}\right\rangle$$
$$=\prod_{i,j}(\tanh\beta)^{(1-P_d^{(i,j)})/2}, \tag{10}$$

where we have introduced the dichotomic link variable

$$P_d^{(i,j)}=\prod_{k\in(i,j)}\tilde{\sigma}_k \tag{11}$$

and $\Sigma_{k\in(i,j)}$ $(\Pi_{k\in(i,j)})$ is the sum (product) running over the $k=1,\ldots,2(d-1)$ plaquettes that have a common link (i,j). The last equality in (10) thus follows from the identity

$$\langle x_{ij}^{\Sigma_{k\in(i,j)}(1+\tilde{\sigma}_k)/2}\rangle=\begin{cases}\langle x_{ij}\rangle=\tanh\beta & \text{if } P_d^{(i,j)}=-1\\ 1 & \text{if } P_d^{(i,j)}=+1.\end{cases}$$

In order to complete the duality transformation, it is convenient to use the variable

$$\tilde{\beta}=-\tfrac{1}{2}\ln\tanh\beta \tag{12}$$

representing the inverse temperature of the dual model. Note that it vanishes as $e^{-2\beta}$ when the temperature $T=\beta^{-1}\to0$. Using (12) and inserting (10) and (9) into (8), one has

$$Z_N=2^{(dN-N_p)}[\tfrac{1}{2}\sinh(2\beta)]^{dN/2}\sum_{\{\tilde{\sigma}_i\}}e^{\tilde{\beta}\sum_{(i,j)}P_d^{(i,j)}}\prod_{i=1}^{N_p}\tilde{J}_i^{(1+\tilde{\sigma}_i)/2}. \tag{13}$$

The quenched free energy (2) thus is

$$-\beta f(\beta)=\frac{d}{2}[(2-d)\ln2+\ln\sinh(2\beta)-(d-1)\tilde{\beta}\tilde{f}(\tilde{\beta})], \tag{14}$$

where the free energy of the dual model is

$$\tilde{f}(\tilde{\beta})=-\lim_{N_p\to\infty}\frac{1}{\tilde{\beta}N_p}\ln\mathscr{Z}_{N_p}, \tag{15}$$

with

$$\mathscr{Z}_{N_p}=\sum_{\{\tilde{\sigma}_i\}}\exp\left(\tilde{\beta}\sum_{(i,j)}P_d^{(i,j)}\right)(-1)^{K(\{\tilde{\sigma}_i\},\{\tilde{J}_i\})} \tag{16}$$

and K is the integer given by

$$K=\sum_{i=1}^{N_P}\left(\frac{1+\tilde{\sigma}_i}{2\pi i}\right)\ln(\tilde{J}_i). \tag{17}$$

Since the Hamiltonian of the dual model can be defined by the relation $\mathscr{Z}_{N_p}=\Sigma_{\{\tilde{\sigma}\}}e^{-\tilde{\beta}H}$, from (16) one sees that

$$H=-\sum_{(i,j)}P_d^{(i,j)}-\sum_{i=1}^{N_p}\ln(\tilde{J}_i)\frac{(1+\tilde{\sigma}_i)}{2\tilde{\beta}}. \tag{18}$$

Let us stress that the nonlinear term of the Hamiltonian (18) describes the interaction of $2d-2$ spins $\tilde{\sigma}$ with a constant coupling instead of two spins σ with a quenched random coupling as happened in the original model. The randomness enters via the presence of a sign. In fact, the weight $\exp(-\tilde{\beta}H)$ does not define a standard Gibbs probability measure on the dual lattice: it defines a signed probability measure. For instance, in two dimensions, $P_d^{(i,j)}=\tilde{\sigma}_i\tilde{\sigma}_j$ so that the Gibbs measure of a configuration of the dual random Ising model differs from that of the pure Ising model only by the presence of a sign determined by the index K related to the correlation between frustrations of the square plaquettes $\{\tilde{J}_i\}$ and configuration $\{\tilde{\sigma}_i\}$.

This is the first result of our paper. Its importance stems from the fact that it is now possible to linearize the Hamiltonian (18) by introducing the magnetization

$$m=\lim_{N_p\to\infty}\frac{1}{N_p}\sum_i^{N_p}\tilde{\sigma}_i. \tag{19}$$

Indeed, if we neglect the fluctuations, the nonlinear term of (18) can be estimated as

$$\sum_{(i,j)}P_d^{(i,j)}=dNm^{2(d-1)}$$

so that (16) becomes

$$\mathscr{Z}_{N_p}=\sum_{\{\tilde{\sigma}\}}\prod_{i=1}^{N_p}\tilde{J}_i^{(1+\tilde{\sigma}_i)/2}\exp(\tilde{\beta}dNm^{2(d-1)}). \tag{20}$$

Let us find the mean-field solution, by using an auxiliary field Ψ. Recalling the saddle point method one immediately sees that in the limit $N\to\infty$,

$$e^{Nd\tilde{\beta}m^{2(d-1)}}\sim\int_{-\infty}^{\infty}d\Phi\,\exp[N\tilde{\beta}(C_dm\Phi^{2d-3}-d\Phi^{2(d-1)})], \tag{21}$$

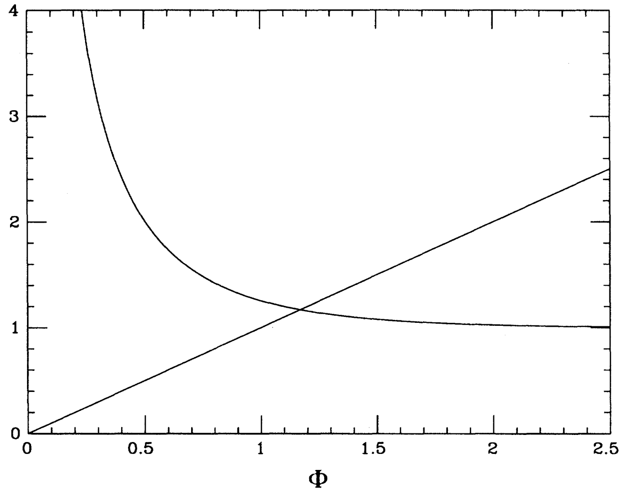

FIG. 1. Graphical solution of the implicit equation (23) in two dimensions, at $T=\beta^{-1}=1$ corresponding to $\tilde{\beta}=0.136\ldots$. The curved line is $\coth(8\tilde{\beta}\Phi)$ versus Φ and the straight line $\Phi=\Phi$.

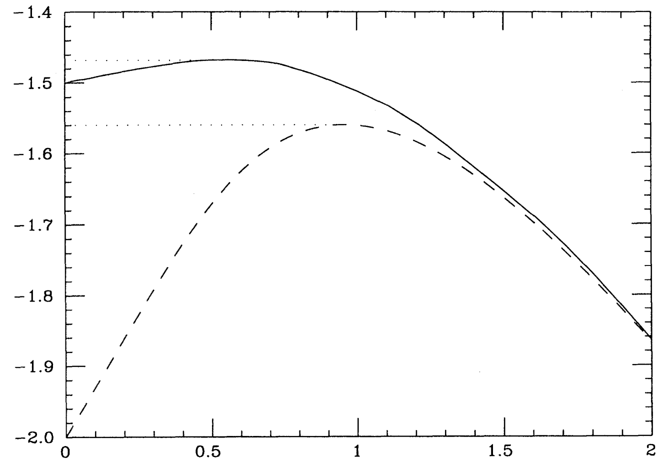

FIG. 2. Random Ising model in two dimensions: the annealed free energy f_a (dashed line) and the mean-field solution (full line) versus temperature $T=\beta^{-1}$. The dotted lines are the Maxwell constructions obtained by imposing that the free energy is a monotonous nondecreasing function of T. One thus obtains the estimates of the ground state energy $E_0 \geqslant -1.559$ (annealed solution) and $E_0 \geqslant -1.468$ (mean-field solution); the numerical result of [5] is $E_0=-1.404\pm0.002$.

where the constant C_d is determined by maximizing the argument of the exponential and reads

$$C_d=2d(d-1)(2d-3)^{-(d-3/2)/(d-1)}. \tag{22}$$

As a consequence one can write the partition function as

$$\mathcal{Z}_{N_p} \sim \max_{\Phi} \exp(-N\beta d\Phi^{2(d-1)}) \times \sum_{\{\tilde{\sigma}\}} \prod_{i=1}^{N_p} \tilde{J}_i^{(1+\tilde{\sigma}_i)/2} e^{\tilde{\beta} C_d \Phi^{2d-3}\tilde{\sigma}_i}. \tag{23}$$

Now we can explicitly carry out the sum over the N_p random variables $\tilde{\sigma}_i$, since they are independent. Indeed, we have to use the identity

$$\sum_{\tilde{\sigma}=\pm1} \tilde{J}_i^{(1+\tilde{\sigma}_i)/2} e^{x\tilde{\sigma}_i} = \begin{cases} 2\cosh(x) & \text{if } \tilde{J}_i=+1 \\ 2\sinh(x) & \text{if } \tilde{J}_i=-1 \end{cases}$$

and remark that, because of the law of large numbers, when $N_p \to \infty$, half of the plaquettes have $\tilde{J}_i=1$ and the other half have $\tilde{J}_i=-1$. As a consequence, since in the thermodynamic limit $-\tilde{\beta}^{-1}\ln(\mathcal{Z}_{N_p})/N_p$ self-averages to the quenched free energy, and $N_p=d(d-1)N/2$, one has

$$-\tilde{\beta}\tilde{f}(\tilde{\beta})=d \max_{\Phi}\left(\frac{(d-1)}{4}\ln[2\sinh(2\tilde{\beta}C_d\Phi^{2d-3})] - \tilde{\beta}\Phi^{2(d-1)}\right). \tag{24}$$

The maximum is realized by the value of Φ^* that is the solution of the self-consistency equation

$$\frac{(2d-3)C_d}{4}\coth(2C_d\tilde{\beta}\Phi^{2d-3})=\Phi. \tag{25}$$

In two dimensions, (25) assumes the simple form $\coth(8\tilde{\beta}\Phi)=\Phi$. The graphical solution of this implicit equation is shown in Fig. 1. One sees that Φ^* should always be larger than unity and at $\tilde{\beta}\to\infty$ (infinite temperature $T=\beta^{-1}$ limit) $\Phi^*=1$. It can appear rather odd that in the dual model the magnetization $\Phi^* \geqslant 1$. This stems from the fact that the Gibbs probability measure $\exp(-\tilde{\beta}H)$ is a signed measure because the random coupling of the original Hamiltonian is transformed into a complex random magnetic field in the dual Hamiltonian (18). From Fig. 1, it is also clear that the mean-field solution does not exhibit phase transitions at finite temperature. However, there is an essential singularity at $T=0$, since inserting (24) into (15) and (14) one sees the $f \sim \exp(1/T)$ for $T\to 0$.

It is important to stress that the mean-field solution does not improve at increasing the dimensionality, since the ratio $N_p/(dN)$ between number of plaquette spins and number of links in the dual lattice increases as $(d-1)$ in contrast to the standard Ising model where the ratio of spins over links decreases as d^{-1}.

It is possible to explicitly solve the self-consistency equation for $\tilde{\beta}\to 0$ where (25) becomes

$$\Phi^* \sim \left(\frac{2d-3}{8}\right)^{1/(2d-2)} \tilde{\beta}^{-1/(2d-2)}. \tag{26}$$

Such a relation shows that when $d\to\infty$ one has $\Phi^*\to 1$ for $T\to 0$ and then for all T's. The high-dimension limit is therefore trivial. The mean-field approximation works at its best in two dimensions. For instance, the zero temperature energy of the mean-field solution is $E_0=-3d/4$ which is a fair estimate in two dimensions where the numerical simulations [5] give $E_0=-1.404\pm0.002$. In Fig. 2, we show the free energy as a function of T in two dimensions compared with the annealed free energy (4). One sees that entropy is negative at low temperature, thus indicating that the solution is unphysical. As a consequence, a better estimate of the ground state energy is given by the maximum of $f(\beta)$, following a standard argument of Toulouse and Vannimenus [6], and one has $E_0 \geq \max_\beta f(\beta) = -1.468$.

In conclusion, we have obtained two main results. (1) We have formulated the random coupling Ising model on the dual lattice made of square plaquettes. The dual model has signed Gibbs probability measure and magnetization larger than unity. (2) We have applied the mean-field approximation to solve the dual model. The approximation is sensible at low dimension.

In our opinion, the mean-field approach on the dual lattice is very promising at least in two and three dimensions. A similar approach has been applied to nondisordered statistical systems with good results [7]. Our method has good heuristic power and there are still many open problems in its framework, such as finding a Ginzburg-Landau criterion or refining the mean-field approximation in a cluster expansion scheme. This might allow one to determine whether there exists a phase transition at low dimension.

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